FREE ENTIRELY ELASTIC OSCILLATORY CHAINS

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PMM Vol.26, No.1, 1962, pp. 172-181

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(Received May 17, 1960)

1. Formulation of the problem. Definition of the concept of oscillatory chains. Let us consider a mechanical system constrained by holonomic, explicitly time independent constraints. Let $q_1, \ldots, q_n$ be Lagrangian coordinates for the system, while $\dot{q}_1, \ldots, \dot{q}_n$ are the corresponding generalized velocities. Assume that the generalized force corresponding to the coordinate q_ν can be expressed in the form

$$Q_\nu(q_1, \ldots, q_n) - R_\nu(\dot{q}_1, \ldots, \dot{q}_n) \qquad (\nu = 1, \ldots, n)$$

Here Q_ν and R_ν are continuous and differentiable functions of their arguments in the region where they are defined. For the given resisting forces we will suppose that for any possible displacement (coinciding with an actual one in the given case) their work is negative

$$-\sum_{\nu=1}^{n} R_\nu(\dot{q}_1, \ldots, \dot{q}_n)\,\dot{q}_\nu < 0 \qquad (1.1)$$

Thereby and from continuity it follows that

$$R_\nu(0, \ldots, 0) = 0 \qquad (\nu = 1, \ldots, n)$$

In the simplest nonlinear case when $R_\nu = f(\dot{q}_\nu)$ $(\nu = 1, \ldots, n)$, condition (1.1) indicates that $a f(a) > 0$ $(a \neq 0)$, while the requirement of continuity indicates, in particular, that $f(0) = 0$. In the linear case conditions (1.1) indicate that the dissipation is complete.

The kinetic energy T of the system will be a quadratic form of generalized coordinates with coefficients depending only on the Lagrangian coordinates in view of the explicit independence of constraints from time

$$T = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(q_1, \ldots, q_n)\,\dot{q}_i\dot{q}_j \qquad (a_{ji} = a_{ij})$$

The equations of motion in the Lagrangian form of second kind will be

$$\sum_{i=1}^{n} a_{\nu i}\ddot{q}_i + \sum_{i,\,j=1} \left(\frac{\partial a_{\nu i}}{\partial q_j} - \frac{1}{2}\frac{\partial a_{ij}}{\partial q_\nu}\right)\dot{q}_i\dot{q}_j = Q_\nu - R_\nu \quad (\nu = 1, \ldots, n) \tag{1.2}$$

We will attempt to investigate the stability in the sense of Liapunov [1,2] of the unperturbed motion

$$q_\nu = q_{\nu 0}(t), \qquad \dot{q}_\nu = \dot{q}_{\nu 0}(t) \qquad (\nu = 1, \ldots, n) \tag{1.3}$$

with respect to the variables $q_1, \ldots, q_r;\ \dot{q}_1, \ldots, \dot{q}_r\ (r \leqslant n)$.

Let us denote the coordinates and velocities for the perturbed motion as

$$q_\nu = q_{\nu 0}(t) + \varkappa_\nu, \qquad \dot{q}_\nu = \dot{q}_{\nu 0}(t) + \dot{\varkappa}_\nu \qquad (\nu = 1, \ldots, n)$$

The differential equations of first approximation for the perturbed motion (equations in variations) can be expressed in the form

$$\sum_{i=1}^{n} (a_{\nu i})_0 \frac{d^2\varkappa_i}{dt^2} + \sum_{i=1}^{n} b_{\nu i}(t)\frac{d\varkappa_i}{dt} + \sum_{i=1}^{n} c_{\nu i}(t)\varkappa_i = 0 \qquad (\nu = 1, \ldots, n) \tag{1.4}$$

where

$$b_{\nu i}(t) = \sum_{j=1}^{n}\left[\left(\frac{\partial a_{\nu j}}{\partial q_i}\right)_0 + \left(\frac{\partial a_{\nu i}}{\partial q_j}\right)_0 - \left(\frac{\partial a_{ij}}{\partial q_\nu}\right)_0\right]\dot{q}_{j0}(t) + \left(\frac{\partial R_\nu}{\partial \dot{q}_i}\right)_0 \tag{1.5}$$

$$c_{\nu i}(t) = \sum_{j=1}^{n}\left\{\left(\frac{\partial a_{\nu j}}{\partial q_i}\right)_0 \ddot{q}_{j0}(t) + \sum_{k=1}^{n}\left[\left(\frac{\partial^2 a_{\nu k}}{\partial q_i \partial q_j}\right)_0 - \frac{1}{2}\left(\frac{\partial^2 a_{jk}}{\partial q_\nu \partial q_i}\right)_0\right]\dot{q}_{j0}(t)\,\dot{q}_{k0}(t) - \left(\frac{\partial Q_\nu}{\partial q_i}\right)_0\right.$$

$$(\nu,\ i = 1, \ldots, n) \tag{1.6}$$

while the index zero in $a_{\nu i}$ and partial derivatives indicates the substitution in them of

$$q_{10}(t), \ldots, q_{n0}(t); \quad \dot{q}_{10}(t), \ldots, \dot{q}_{n0}(t)$$

Let us refer to the original mechanical system as the "oscillatory chain" with respect to the unperturbed motion (1.3) if it is possible to choose such Lagrangian coordinates for which the coefficients $(a_{\nu i})_0$, $b_{\nu i}(t)$ and $c_{\nu i}(t)$ are such that for some natural $m < n$

$$(a_{\nu i})_0 = 0 \tag{1.7}$$

$$(\nu = 1, \ldots, m;\ i = m+1, \ldots, n$$
$$\nu = m+1, \ldots, n;\ i = 1, \ldots, m)$$

$$b_{\nu i}(t) = (\partial R_\nu)/\partial \dot{q}_i)_0 \tag{1.8}$$

$$(\nu = 1, \ldots, n;\ i = 1, \ldots, n)$$

$$(\partial R_\nu/\partial \dot{q}_i)_0 = 0 \tag{1.9}$$

$$(\nu = 1, \ldots, m;\ i = m+1, \ldots, n$$
$$\nu = m+1, \ldots, n;\ i = 1, \ldots, m)$$

$$c_{\nu i}(t) = 0$$
$$(\nu = 1, \ldots, m;\ i = m+1, \ldots, n \quad \nu = m+1, \ldots, n;\ i = 1, \ldots, m) \qquad (1.10)$$

for all t larger than some t_0. Conditions (1.7) to (1.10) indicate that the matrix functions of the coefficients for system (1.4) are of the form

$$\begin{pmatrix} \| (a_{\nu i})_0 \|_1^m & 0 \\ 0 & \| (a_{\nu i})_0 \|_{m+1}^n \end{pmatrix}$$

$$\begin{pmatrix} \left\| \left(\frac{\partial R_\nu}{\partial \dot{q}_i} \right)_0 \right\|_1^m & 0 \\ 0 & \left\| \left(\frac{\partial R_\nu}{\partial \dot{q}_i} \right)_0 \right\|_{m+1}^n \end{pmatrix}$$

$$\begin{pmatrix} \| c_{\nu i}(t) \|_1^m & 0 \\ 0 & \| c_{\nu i}(t) \|_{m+1}^n \end{pmatrix}$$

Upon fulfillment of conditions (1.7) to (1.10) the equations in variations (1.4) divide into two groups of m and $n - m$ equations

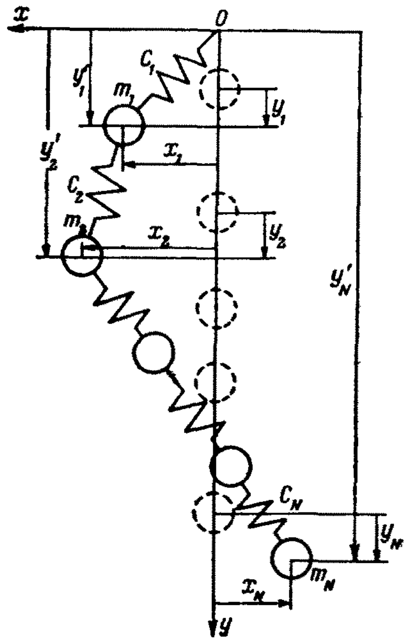

Fig. 1.

$$\sum_{i=1}^{m} (a_{\nu i})_0 \frac{d^2 \varkappa_i}{dt^2} + \sum_{i=1}^{m} \left(\frac{\partial R_\nu}{\partial \dot{q}_i} \right)_0 \frac{d\varkappa_i}{dt} + \sum_{i=1}^{m} c_{\nu i}(t) \varkappa_i = 0 \qquad (\nu = 1, \ldots, m) \qquad (1.11)$$

$$\sum_{i=m+1}^{n} (a_{\nu i})_0 \frac{d^2 \varkappa_i}{dt^2} + \sum_{i=m+1}^{n} \left(\frac{\partial R_\nu}{\partial \dot{q}_i} \right)_0 \frac{d\varkappa_i}{dt} + \sum_{i=m+1}^{n} c_{\nu i}(t) \varkappa_i = 0 \quad (\nu = m+1, \ldots, n) \quad (1.12)$$

2. Determination of the equilibrium position for a free entirely elastic oscillatory chain. The simplest example of an "oscillatory chain" is a *free entirely elastic oscillatory chain with respect to vertical oscillations* (i.e. when the unperturbed motion is a vertical oscillation of the referred system) (see for example [3]). Figure 1 shows a system of N material points with masses $m_1, \ldots, m_N$ sequentially connected by N springs (the mass of which is neglected) with stiffnesses $c_1, \ldots, c_N$ and the lengths in the unstressed state $l_1, \ldots, l_N$. The beginning of the first spring is attached at point O while the beginnings of each of the following springs are attached to weightless hinges with axes perpendicular to the vertical surface Oxy, thus producing plane motion.

By this means the system is constrained by only N trivial constraints: $z_1 = 0, \ldots, z_N = 0$ while the $2N$ Lagrangian coordinates are $x_1, \ldots, x_N$;

$y_1', \ldots, y_N'$ are the rectangular coordinates of the material points $m_1, \ldots, m_N$. The kinetic energy for this simplest case is

$$T = \frac{1}{2} \sum_{k=1}^{N} m_k (\dot{x}_k^2 + \dot{y}_k'^2)$$

i.e. $a_{ij} = m_i \delta_{ij}$ (δ_{ij} is the Kronecker delta; $i, j = 1, \ldots, N$). Let us compute the potential energy $V(x, \ldots, x_N; y_1'; \ldots, y_N')$ for the linear forces of spring elasticity and gravity

$$V = -g \sum_{k=1}^{N} m_k y_k' + \frac{1}{2} \sum_{k=1}^{N} c_k [(x_k - x_{k-1})^2 + (y_k' - y_{k-1}')^2] - \\ - \sum_{k=1}^{N} c_k l_k \sqrt{(x_k - x_{k-1})^2 + (y_k' - y_{k-1}')^2} \tag{2.1}$$

considering $x_0 = y_0' = 0$ and taking only the arithmetic value of the root. We will start with the determination of system's position of equilibrium for which we will consider the system

$$\frac{\partial V}{\partial x_k} = -c_k x_{k-1} + (c_k + c_{k+1}) x_k - c_{k+1} x_{k+1} - c_k l_k [(x_k - x_{k-1})^2 + \\ + (y_k' - y_{k-1}')^2]^{-1/2} (x_k - x_{k-1}) + c_{k+1} l_{k+1} [(x_{k+1} - x_k)^2 + (y_{k+1}' - y_k')^2]^{-1/2} \times \\ \times (x_{k+1} - x_k) = 0 \quad (k=1, \ldots, N) \tag{2.2}$$

$$\frac{\partial V}{\partial y_k'} = -m_k g - c_k y_{k-1}' + (c_k + c_{k+1}) y_k' - c_{k+1} y_{k+1}' - c_k l_k [(x_k - x_{k-1})^2 + \\ + (y_k' - y_{k-1}')^2]^{-1/2} (y_k' - y_{k-1}') + c_{k+1} l_{k+1} [(x_{k+1} - x_k)^2 + (y_{k+1}' - y_k')^2]^{-1/2} \times \\ \times (y_{k+1}' - y_k') = 0 \quad (k=1, \ldots, N)$$

with $c_{N+1} = l_{N+1} = 0$. This system possesses the solution (lower position of equilibrium)

$$x_k = 0, \qquad y_k' = (l_1 + \lambda_1) + \ldots + (l_k + \lambda_k) \qquad (k = 1, \ldots, N) \tag{2.3}$$

where λ_j denotes the static elongation of the jth spring

$$\lambda_j = (m_j + m_{j+1} + \ldots + m_N) g / c_j \qquad (j = 1, \ldots, N)$$

The found position of equilibrium will be isolated. Indeed, the Jacobian for the system of Equations (2.2) for values of the variables (2.3) $D = D_1 D_2$ where D_1 and D_2 are the determinants for Jacobian matrices of Nth order

$$D_1 = \begin{vmatrix} \frac{c_1\lambda_1}{l_1+\lambda_1} + \frac{c_2\lambda_2}{l_2+\lambda_2} & -\frac{c_2\lambda_2}{l_2+\lambda_2} & 0 & \cdots & 0 \\ -\frac{c_2\lambda_2}{l_2+\lambda_2} & \frac{c_2\lambda_2}{l_2+\lambda_2} + \frac{c_3\lambda_3}{l_3+\lambda_3} & -\frac{c_3\lambda_3}{l_3+\lambda_3} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \frac{c_N\lambda_N}{l_N+\lambda_N} \end{vmatrix}$$

while D_2 is obtained from D_1, for $l_k = 0$, $\lambda_k = 1$ ($k = 1, \ldots, N$). Having utilized the formula for the determinant of the Jacobian matrix we find

$$D = \prod_{k=1}^{N} \frac{c_k^2\lambda_k}{l_k+\lambda_k} > 0$$

which is the required result.

Let us introduce the variables y_k, which represent deviations along the vertical of the kth material point from the lower equilibrium position

$$y_k = y_k' - (l_1+\lambda_1) - \ldots - (l_k+\lambda_k) \qquad (k = 1, \ldots, N) \tag{2.4}$$

At the lower position of equilibrium $x_1 = \ldots = x_N = y_1 = \ldots = y_N = 0$. For determination of positions of equilibrium other than the lowest one, we will write the system of Equations (2.2) in the form

$$\begin{gathered} -c_k(x_k - x_{k-1})\{l_k[(x_k - x_{k-1})^2 + (l_k+\lambda_k+y_k-y_{k-1})^2]^{-1/2} - 1\} + \\ + c_{k+1}(x_{k+1}-x_k)\{l_{k+1}[(x_{k+1}-x_k)^2 + (l_{k+1}+\lambda_{k+1}+y_{k+1}-y_k)^2]^{-1/2} - 1\} = 0 \qquad (2.5) \\ -c_k(l_k+\lambda_k+y_k'-y_{k-1})\{l_k[(x_k-x_{k-1})^2 + (l_k+\lambda_k+y_k-y_{k-1})^2]^{-1/2} - 1\} + \\ + c_{k+1}(l_{k+1}+\lambda_{k+1}+y_{k+1}-y_k)\{l_{k+1}[(x_{k+1}-x_k)^2 + \\ + (l_{k+1}+\lambda_{k+1}+y_{k+1}-y_k)^2]^{-1/2} - 1\} = m_k g \quad (k = 1, \ldots, N) \end{gathered}$$

Let us write the equations corresponding to k equal N

$$\begin{gathered} c_N(x_N - x_{N-1})\{1 - l_N[(x_N-x_{N-1})^2 + (l_N+\lambda_N+y_N-y_{N-1})^2]^{-1/2}\} = 0 \\ c_N(l_N+\lambda_N+y_N-y_{N-1})\{1 - l_N[(x_N-x_{N-1})^2 + (l_N+\lambda_N+y_N-y_{N-1})^2]^{-1/2}\} = m_N g \end{gathered}$$

The braced terms are not zero, because otherwise the second equation could not be fulfilled. The first equation yields $x_N = x_{N-1}$ while the second equation becomes

$$c_N(l_N+\lambda_N+y_N-y_{N-1})\left\{1 - \frac{l_N}{|l_N+\lambda_N+y_N-y_{N-1}|}\right\} = m_N g$$

This equation always has a solution $y_N = y_{N-1}$, while if $\lambda_N < l_N$ it also has a solution $y_N = y_{N-1} - 2l_N$. Let us consider the most common case when the static elongation of each spring is less than its length in the unstressed condition

$$\lambda_k < l_k \qquad (k = 1, \ldots, N) \tag{2.6}$$

Treating Equations (2.5) corresponding to $k = N - 1,\ N - 2,\ \ldots,\ 1$ analogously, we will find 2^N equilibrium positions of a free oscillatory chain: $x_1 = \ldots = x_N = 0$

$$y_1 = \begin{cases} 0, \\ -2l_1, \end{cases} \qquad y_2 = \begin{cases} y_1, \\ y_1 - 2l_2, \end{cases} \quad \ldots, y_N = \begin{cases} y_{N-1} \\ y_{N-1} - 2l_N \end{cases} \tag{2.7}$$

It must be noted, however, that the planes of motion for each of the N material points are different and parallel to the vertical plane.

3. Asymptotic stability in the large of the lower equilibrium position in the presence of resistance forces. The equations of motion (1.2) for the free entirely elastic oscillatory chain are expressed quite simply as

$$\begin{aligned} m_k\ddot{x}_k &= -\frac{\partial V}{\partial x_k} - R_k(\dot{x}_1, \ldots, \dot{x}_N; \dot{y}_1, \ldots, \dot{y}_N) \\ m_k\ddot{y}_k &= -\frac{\partial V}{\partial y_k} - R_{N+k}(\dot{x}_1, \ldots, \dot{x}_N; \dot{y}_1, \ldots, \dot{y}_N) \end{aligned} \qquad (k = 1, \ldots, N) \tag{3.1}$$

Let us indicate by $\inf V$ the smallest value of the potential energy for a free entirely elastic oscillatory chain among $(2^N - 1)$ positions of equilibrium different from the lowest one. In the phase space $x_1, \ldots, x_N;\ y_1, \ldots, y_N;\ \dot{x}_1, \ldots, \dot{x}_N,\ \dot{y}_1, \ldots, \dot{y}_N$ we define a closed region G_0 by the inequality

$$T + V \leqslant \inf V$$

Theorem. In the presence of resistance forces satisfying condition (1.1), the lower position of equilibrium for a free entirely elastic oscillatory chain is asymptotically stable for initial deviations $x_1^{(0)}, \ldots, x_N^{(0)};\ y_1^{(0)}, \ldots, y_N^{(0)};\ \dot{x}_1^{(0)}, \ldots, \dot{x}_N^{(0)};\ \dot{y}_1^{(0)}, \ldots, \dot{y}_N^{(0)}$ within the region G. This indicates that $T^{(0)} + V^{(0)}$ satisfies the inequality

$$T^{(0)} + V^{(0)} < \inf V \tag{3.2}$$

where substitutions $x = x_1^{(0)}, \ldots, y_N = y_N^{(0)}$ were made in $T^{(0)}$ and

$V^{(0)}$.

Proof. Let us take the total energy of the system as the function v of Theorem 14.1 in [4]

$$v = T + V - V(0, \ldots, 0) \tag{3.3}$$

Compute $V(0, \ldots, 0)$, i.e. the value of potential energy at the lower equilibrium position

$$V(0, \ldots, 0) = -g \sum_{k=1}^{N} m_k [(l_1 + \lambda_1) + \ldots + (l_k + \lambda_k)] - \frac{1}{2} \sum_{k=1}^{N} c_k (l_k^2 - \lambda_k^2)$$

We will show that $V - V(0, \ldots, 0)$ will be a positive definite function of $x_1, \ldots, x_N; y_1, \ldots, y_N$ in the sense of Liapunov. Let us transform $V - V(0, \ldots, 0)$ into the form

$$V - V(0, \ldots, 0) = \frac{1}{2} \sum_{k=1}^{N} c_k [(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2 + 2l_k (l_k + \lambda_k + y_k - y_{k-1}) - \\ - 2l_k \sqrt{(x_k - x_{k-1})^2 + (l_k + \lambda_k + y_k - y_{k-1})^2}]$$

and establish the validity of inequalities

$$(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2 + 2l_k (l_k + \lambda_k + y_k - y_{k-1}) \geqslant \\ \geqslant 2l_k \sqrt{(x_k - x_{k-1})^2 + (l_k + \lambda_k + y_k - y_{k-1})^2} \tag{3.4}$$

($k = 1, \ldots, N$; $x_0 = y_0 = 0$). The right-hand side of the inequalities can be expressed as

$$(x_k - x_{k-1})^2 + (y_k - y_{k-1} + l_k)^2 + l_k (l_k + 2\lambda_k)$$

and is obviously positive. Let us square the inequalities (3.4) where upon transformation we obtain

$$[(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2 + 2l_k (y_k - y_{k-1})]^2 + \\ + 4l_k\lambda_k [(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2] \geqslant 0 \qquad (k = 1, \ldots, N)$$

The derived inequalities are valid and the simultaneous existence of the equality sign is possible only for $x_1 = \ldots = x_N = y_1 = \ldots = y_N = 0$ Consequently, the total energy (3.3) of the system will be a positive definite function of all Lagrangian coordinates and velocities in the Liapunov sense. Its derivative, on the strength of Equation (3.1), is

$$\frac{d}{dt} [T + V - V(0, \ldots, 0)] = - \sum_{k=1}^{N} (R_k \dot{x}_k + R_{N+k} \dot{y}_k) \leqslant 0$$

In view of definition of the considered resistance forces, the equating to zero in the last inequality is possible only for the position of equilibrium. A motion which started in the G domain cannot leave it, and in it the position of equilibrium will be unique. Thus the conditions of Theorem 14.1 of [4] are fulfilled. The theorem is proved.

Note. It is possible to establish formulas for the radius of a sphere or the edge of a cube inscribed in a closed $4N$ dimensional region G_0.

4. Equations in variations for vertical oscillations of the system. Let us write out in detail Equations (3.1)

$$\begin{aligned} m_k\ddot{x}_k &= -c_k(x_k - x_{k-1}) + c_{k+1}(x_{k+1} - x_k) + c_k l_k (x_k - x_{k-1})[(x_k - x_{k-1})^2 + \\ &\quad + (l_k + \lambda_k + y_k - y_{k-1})^2]^{-1/2} - c_{k+1} l_{k+1}(x_{k+1} - x_k)[(x_{k+1} - x_k)^2 + \\ &\quad + (l_{k+1} + \lambda_{k+1} + y_{k+1} - y_k)^2]^{-1/2} - R_k(\dot{x}_1, \ldots, \dot{x}_N;\ \dot{y}_1, \ldots, \dot{y}_N) \\ m_k\ddot{y}_k &= -c_k l_k + c_{k+1} l_{k+1} - c_k(y_k - y_{k-1}) + c_{k+1}(y_{k+1} - y_k) + \\ &\quad + c_k l_k (l_k + \lambda_k + y_k - y_{k-1})[(x_k - x_{k-1})^2 + (l_k + \lambda_k + y_k - y_{k-1})^2]^{-1/2} - \\ &\quad - c_{k+1} l_{k+1}(l_{k+1} + \lambda_{k+1} + y_{k+1} - y_k)[(x_{k+1} - x_k)^2 + (l_{k+1} + \lambda_{k+1} + y_{k+1} - y_k)^2]^{-1/2} - \\ &\quad - R_{N+k}(\dot{x}_1, \ldots, \dot{x}_N;\ \dot{y}_1, \ldots, \dot{y}_N) \qquad (k = 1, \ldots, N) \end{aligned} \tag{4.1}$$

Let us assume that the projections on the x axis of the resisting forces satisfy the conditions

$$R_k(0, \ldots, 0;\ \dot{y}_1, \ldots, \dot{y}_N) = 0 \qquad (k = 1, \ldots, N)$$

System (4.1) possesses the solution (unperturbed motion (1.3))

$$x_k = x_{k0}(t) \equiv 0, \qquad y_k = y_{k0}(t) \qquad (k = 1, \ldots, N) \tag{4.2}$$

Here $y_{k0}(t)$ satisfy the system of equations

$$\begin{aligned} \ddot{y}_{k0} + \frac{1}{m_k} R_{N+k}(0, \ldots, 0;\ \dot{y}_{10}, \ldots, \dot{y}_{N0}) - p_k y_{k-1,0} + \\ + (p_k + \mu_{k+1} p_{k+1}) y_{k0} - \mu_{k+1} p_{k+1} y_{k+1,0} = 0 \qquad (k = 1, \ldots, N) \end{aligned} \tag{4.3}$$

and

$$p_k = \frac{c_k}{m_k} \quad (k = 1, \ldots, N;\ p_{N+1} = 0), \qquad \mu_k = \frac{m_k}{m_{k-1}} \quad (k = 2, \ldots, N;\ \mu_{N+1} = 0)$$

Let us verify the fulfillment of conditions (1.7) to (1.10) ($n = 2N$, $s = N$). Conditions (1.7) and (1.8) are satisfied since

$$a_{\nu i} = \begin{cases} m_\nu \delta_{\nu i} & (\nu = 1, \ldots, N;\ i = 1, \ldots, 2N) \\ m_{\nu - N}\delta_{\nu i} & (\nu = N+1, \ldots, 2N;\ i = 1, \ldots, 2N) \end{cases}$$

Condition (1.9) requires that

$$\left(\frac{\partial R_k}{\partial \dot{y}_l}\right)_0 = 0, \qquad \left(\frac{\partial R_{N+k}}{\partial \dot{x}_l}\right)_0 = 0 \qquad (k, l = 1, \ldots, N) \tag{4.4}$$

where the zero index denotes that after differentiation the values of the arguments from (4.2) are substituted. Conditions (4.4) will be satisfied, in particular, if R_k are independent from $\dot{y}_l$, and R_{N+k} from $\dot{x}_l$ (k, l = 1, ..., N). We shall assume the conditions (4.4) to be fulfilled.

Condition (1.10) requires that

$$\left(\frac{\partial^2 V}{\partial x_j \partial y_k}\right)_0 = 0 \qquad (j, k = 1, \ldots, N)$$

which is fulfilled, as suffix 0 signifies, in particular, that after differentiation assumed $x_1 = \ldots = x_N = 0$. Consequently, the equations in variations (1.11) and (1.12) take place for the perturbed motion ($X_k = 0 + \xi_k$, $y_k = y_{k0}(t) + \eta_k$; k = 1, ..., N)

$$\frac{d^2\xi_k}{dt^2} + \frac{1}{m_k}\sum_{i=1}^{N}\left(\frac{\partial R_k}{\partial \dot{x}_i}\right)_0 \frac{d\xi_i}{dt} + \frac{1}{m_k}\sum_{i=1}^{N}\left(\frac{\partial^2 V}{\partial x_i \partial x_k}\right)_0 \xi_i = 0$$

$$\frac{d^2\eta_k}{dt^2} + \frac{1}{m_k}\sum_{i=1}^{N}\left(\frac{\partial R_{N+k}}{\partial \dot{y}_i}\right)_0 \frac{d\eta_i}{dt} + \frac{1}{m_k}\sum_{i=1}^{N}\left(\frac{\partial^2 V}{\partial y_i \partial y_k}\right)_0 \eta_i = 0$$

or in the expanded form

$$\frac{d^2\xi_k}{dt^2} + \frac{1}{m_k}\sum_{i=1}^{N}\left(\frac{\partial R_k}{\partial \dot{x}_i}\right)_0 \frac{d\xi_i}{dt} - p_k\left\{1 - \left[1 + \gamma_k + \frac{(y_{k0}(t) - y_{k-1,0}(t))}{l_k}\right]^{-1}\right\}(\xi_{k-1} -$$

$$- \xi_k) + \mu_{k+1} p_{k+1}\left\{1 - \left[1 + \gamma_{k+1} + \frac{(y_{k+1,0}(t) - y_{k0}(t))}{l_{k+1}}\right]^{-1}\right\}(\xi_k - \xi_{k+1}) = 0 \qquad (4.5)$$

$$\frac{d^2\eta_k}{dt^2} + \frac{1}{m_k}\sum_{i=1}^{N}\left(\frac{\partial R_{N+k}}{\partial \dot{y}_i}\right)_0 \frac{d\eta_i}{dt} - p_k\eta_{k-1} + (p_k + \mu_{k+1} p_{k+1})\eta_k - \mu_{k+1} p_{k+1}\eta_{k+1} = 0$$

$$(k = 1, \ldots, N) \qquad (4.6)$$

$$\gamma_k = \lambda_k / l_k \qquad (k = 1, \ldots, N;\ \gamma_{N+1} = 0)$$

Let us note that these equations are derived despite the fact that the question of stability in the large for the lower position of equilibrium during fulfillment of condition (1.1) is solved by the theorem of Section 3. One considers, first, the case when condition (1.1) is not fulfilled (for example, during partial dissipation), second, use of Equations (4.5) to (4.6) for determination of stability of unperturbed motion (4.2), and third, the absence of resistance forces. This conservative case is treated next.

5. Conservative case. In the absence of resisting forces, the free, entirely elastic oscillatory chain is a conservative system. Deviations $y_{k0}(t)$ of its masses from the lower position of equilibrium during vertical oscillations (unperturbed motion) satisfy the System (4.3) for $R_{N+k} \equiv 0$ ($k = 1, \ldots, N$), describing small oscillations of Sturm systems [5]. The equation of frequencies ω for this system is a characteristic equation of a certain Jacobian matrix.

$$\begin{vmatrix} \omega^2 - (p_1 + \mu_2 p_2) & \mu_2 p_2 & 0 \; \ldots & 0 \\ \mu_2 p_2 & \omega^2 - (p_2 + \mu_3 p_3) & \mu_3 p_3 \; \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 \; \ldots & \omega^2 - p_N \end{vmatrix} = 0$$

Equations (4.5) from the system of equations for the first approximation of perturbed motion will have for $R_j \equiv 0$ ($j = 1, \ldots, 2N$) either periodic coefficients in case of commensurate frequencies $\omega_1, \ldots, \omega_N$, or almost periodic in the opposite case. The stability investigation of unperturbed motion is in both cases a considerably difficult problem.

The investigation is facilitated by the fact that in the conservative case all solutions of system (4.6) are bounded. This follows from the positiveness of the eigenvalues for the above written Jacobian matrix. The bounded nature of solutions can also be established directly by writing the system (4.6) with $R_{N+k} \equiv 0$ ($k = 1, \ldots, N$) in the form

$$\frac{d\eta_k}{dt} = \dot{\eta}_k, \quad \frac{d\dot{\eta}_k}{dt} = p_k \eta_{k-1} - (p_k + \mu_{k+1} p_{k+1}) \eta_k + \mu_{k+1} p_{k+1} \eta_{k+1} \quad (k = 1, \ldots, N)$$

Consider now the positive definite quadratic form of the variables $\eta_1, \ldots, \eta_N$; $\dot{\eta}_1, \ldots, \dot{\eta}_N$ with constant coefficients

$$U = \frac{1}{2} \sum_{k=1}^{N} \mu_1 \ldots \mu_k [p_k (\eta_k - \eta_{k-1})^2 + \dot{\eta}_k^2] \qquad (\mu_1 = 1, \ \eta_0 = 0)$$

In view of the given equations, the derivative of the quadratic form is equal to zero, which establishes the bounded nature of the solutions.

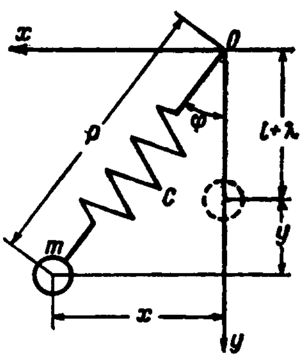

Fig. 2.

Examples: A single-link, free and entirely elastic oscillatory chain is represented by a mathematical pendulum of mass m with a spring of stiffness c and unstressed length l (Fig. 2). The system of Equations (4.3) is reduced to one equation

$$m\ddot{y}_0 + cy_0 = 0$$

where for the unperturbed motion we will have

$$x \equiv 0, \quad y = y_0(t) = Y \cos \omega t \qquad \left(\omega = \sqrt{\frac{c}{m}}\right)$$

The equations in variations (4.5) and (4.6) will be expressed as

$$\frac{d^2\xi}{dt^2}+\omega^2\left[1-\frac{1}{1+\gamma+Y\cos\omega t}\right]\xi=0, \qquad \frac{d^2\eta}{dt^2}+\omega^2\eta=0 \tag{5.1}$$

Note that the fulfillment of condition (1.8) is not an intrinsic property of the mechanical system itself and of the chosen unperturbed motion, but is determined also by the choice of the Lagrangian coordinates. Using polar coordinates in the example considered, we obtain

$$T=\frac{1}{2}m(\rho^2\dot{\varphi}^2+\dot{\rho}^2), \qquad V=\frac{1}{2}c(\rho-l)^2-mg\rho\cos\varphi$$

and, in the absence of resisting forces, the Lagrange equations become

$$2\,\rho\,\dot{\rho}\,\dot{\varphi}+\rho^2\ddot{\varphi}=-g\rho\sin\varphi, \qquad \ddot{\rho}-\rho\dot{\varphi}^2=-\frac{c}{m}(\rho-l)+g\cos\varphi$$

The vertical oscillation of the mass m or unperturbed motion now becomes in previous notation

$$\varphi=\varphi_0\equiv 0, \qquad \rho=\rho_0(t)=l+\lambda+Y\cos\omega t$$

Formula (1.5) yields for the coefficient $b_{11}(t)$

$$b_{11}(t)=2m\rho_0\dot{\rho}_0\not\equiv 0$$

which indicates violation of condition (1.8). Equations in variations for the perturbed motion ($\phi = 0 + \Phi$, $\rho = \rho_0(t) + P$ will assume the following form in polar coordinates

$$\frac{d^2\Phi}{d\tau^2}-2\frac{a\sin\tau}{1+\gamma+a\cos\tau}\frac{d\Phi}{d\tau}+\frac{\gamma}{1+\gamma+a\cos\tau}\Phi=0$$

$$\frac{d^2P}{d\tau^2}+P=0 \qquad \left(\gamma=\frac{\lambda}{l},\ a=\frac{Y}{l},\ \tau=\omega t\right)$$

As we can see, the appearance of a first derivative term in the equation in variations is possible also in a conservative system.

Returning to Cartesian coordinates and introducing nondimensional time $\tau = \omega t$, we write down the differential equation of perturbed motion in the form

$$\frac{d^2\xi}{d\tau^2}+\frac{\gamma+a\cos\tau}{1+\gamma+a\cos\tau}\xi+H(\tau,\xi,\eta)=0, \qquad \frac{d^2\eta}{d\tau^2}+\eta+\chi(\tau,\xi,\eta)=0$$

Here, as well as above, γ and a are nondimensional parameters expressing the ratios of static elongation and amplitude of vertical oscillation to nondeformed length of the spring

$$\gamma=\frac{\lambda}{l}, \qquad a=\frac{Y}{l}$$

Stability or instability of a trivial solution of equations in variations (5.1) is determined as such for the first of Equations (5.1). Nevertheless, in the considered conservative case, the stability of the trivial solution of (5.1) does not at all determine the stability of unperturbed motion with respect to the variables x, y, $\dot{x}$, $\dot{y}$; since one of the critical cases is involved. The instability, however, of the trivial solution of system (5.1) involves (with possibly the exception of boundary cases) the instability of unperturbed motion ([2, p.70]) with respect to the variables x, y; $\dot{x}$, $\dot{y}$.

This is dependent on the fact that the first equation in variations has as its coefficient a periodic function, and when its trivial solution is unstable the lowest characteristic number in the sense of Liapunov is negative.

In considering stability of the trivial solution of equation

$$\frac{d^2\xi}{d\tau^2}+\frac{\gamma+a\cos\tau}{1+\gamma+a\cos\tau}\xi=0 \tag{5.2}$$

we begin with the Zhukovskii [6] criterion which assures stability when the following inequalities are fulfilled

$$\frac{k^2}{4}\leqslant p(t)\leqslant\frac{(k+1)^2}{4} \qquad (k=0,\ 1,\ 2,\ldots)$$

For $a < 1 + \gamma$, expressing a natural limit that the amplitude of longitudinal oscillations is less than static deformation of the spring we have

$$\inf p(\tau)=\frac{\gamma-a}{1+\gamma-a}, \qquad \sup p(\tau)=\frac{\gamma+a}{1+\gamma+a}$$

The Zhukovskii criterion requires the fulfillment of either inequalities

$$a\leqslant\gamma, \qquad a\leqslant\frac{1}{3}-\gamma \qquad (\text{for } k=0)$$

or inequality

$$a\leqslant-\frac{1}{3}+\gamma \qquad (\text{for } k=1)$$

For $k > 1$ the criterion fails.

The resulting stability region for the trivial solution of Equation (5.2), obtainable from the Zhukovskii criterion, is shown crosshatched in Fig. 3. This plot will be useful for comparison with the region of instability.

In finding the region of instability by the method of small parameters [7,8], let us take a as a small parameter and write down the Equation

(5.2) in the form

$$\frac{d^2\xi}{d\tau^2} + [p_0(\gamma) + ap_1(\tau, \gamma) + a^2p_2(\tau, \gamma) + \ldots]\,\xi = 0$$

where

$$p_0(\gamma) = \frac{\gamma}{1+\gamma}, \qquad p_1(\tau, \gamma) = 2p_1^{(1)}(\gamma)\cos\tau \qquad \left(p_1^{(1)}(\gamma) = \frac{1}{2(1+\gamma)^2}\right)$$

In the scalar case the regions of instability in the $a\gamma$ plane can touch the $a = 0$ axis at those points γ_m which are the roots of the equation

$$2\sqrt{p_0(\gamma_m)} = m \quad \text{or} \quad \gamma_m = \frac{m^2}{4 - m^2} \qquad (m = 1, 2, \ldots)$$

For $\gamma > 0$ a wide region of instability (i.e. with nonzero angle between the tangents) touches only at the point $\gamma_1 = 1/3$ with no other such points on the $\gamma > 0$ axis existing. Tangent of the slope for the tangential line will be determined in our example from the formula resulting from Formula (6) [8]

$$\chi^{\pm} = \pm\left[\frac{p_1^{(1)}(\gamma)}{dp_0/d\gamma}\right]_{\gamma=\gamma_1} = \pm\frac{1}{2}\,.$$

This establishes, in the first approximation, the instability region for vertical oscillations of a pendulum on a spring

$$\tfrac{1}{3} - \tfrac{1}{2}a + \ldots < \gamma < \tfrac{1}{3} + \tfrac{1}{2}a + \ldots$$

Rays bounding this region are shown by dotted lines in Fig. 3.

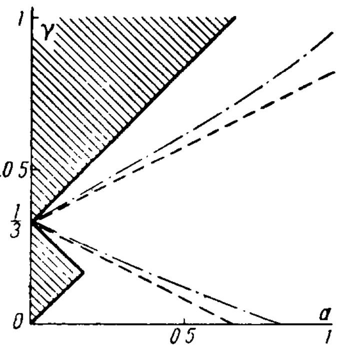

Fig. 3.

From the general theory it follows that since

$$\left.\frac{dp_0}{d\gamma}\right|_{\gamma=\gamma_1} \neq 0$$

the equations of the boundaries will be analytic functions of the parameter a, therefore, the rejected terms are not of lower order than a^2. The remaining coefficients of the expansion can be computed utilizing the fact that on the boundary of this region of instability there exists an antiperiodic (inasmuch as m is odd) solution. We will give the final result: in the second approximation the region of instability is determined from the inequalities

$$\tfrac{1}{3} - \tfrac{1}{2}a + \tfrac{15}{128}a^2 + \ldots < \gamma < \tfrac{1}{3} + \tfrac{1}{2}a + \tfrac{15}{128}a^2 + \ldots$$

The curves bounding this region are shown dash-dotted in Fig. 3.

It is known from experiments [3] that the perturbation of vertical oscillations in the presence of resisting forces takes place for the pendulum on a spring with $\lambda \approx (1.3)l$, i.e. for $\gamma \approx 1/3$. The investigation of the transfer of longitudinal oscillations into transverse ones was outlined by Mettler [3]. He utilizes the method of slowly varying amplitude and phase suggested by Bogoliubov and Krylov [9] and developed by Mitropol'ski [10].

Conclusion. Regions of "conservative instability" will generate in the dissipative case regions of instability of vertical oscillations. Notwithstanding asymptotic damping of oscillations, which for not very large dissipation will occur slowly, practically large variations of oscillations due to autoresonance in the chain can be quite substantial for evaluation of system performance.

I am grateful to the LGU aspirant B.G. Pittel' who performed the calculations for the example.

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Translated by V.C.